

Modulated electromagnetic fields in inhomogeneous media, hyperbolic pseudoanalytic functions and transmutations

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October 21, 2014

Abstract

The time-dependent Maxwell system describing electromagnetic wave propagation in inhomogeneous isotropic media in the one-dimensional case reduces to a Vekua-type equation for bicomplex-valued functions of a hyperbolic variable, see [7]. Using this relation we solve the problem of the transmission through an inhomogeneous layer of a normally incident electromagnetic time-dependent plane wave. The solution is written in terms of a pair of Darboux-associated transmutation operators [9], and combined with the recent results on their construction [10], [11] can be used for efficient computation of the transmitted modulated signals. We develop the corresponding numerical method and illustrate its performance with examples.

1 Introduction

In the present work a method for solving the problem of time-dependent electromagnetic wave propagation through an isotropic inhomogeneous medium is developed. Several ideas concerning such mathematical notions as bicomplex and biquaternionic reformulations of electromagnetic models, hyperbolic Vekua-type equations and transmutation operators from the theory of ordinary linear differential equations are combined in our approach which results in a simple and practical representation for solutions.

We observe that the 1+1 Maxwell system for inhomogeneous media can be transformed into a hyperbolic Vekua equation. This gives us the possibility to obtain the exact solution of the problem of a normally incident electromagnetic time-dependent plane wave propagated through an inhomogeneous layer in terms of a couple of Darboux-associated transmutation operators. This is a new representation for a solution of the classic problem. Application of the recent results on the analytic approximation of such operators allows us to write down the electromagnetic wave in an approximate analytic form which is then used for numerical computation. The numerical implementation of the proposed approach reduces to a certain recursive integration and solution of

*The authors acknowledge the support from CONACYT, Mexico via the projects 166141 and 222478.

an approximation problem, and can be based on the usage of corresponding standard routines of such packages as Matlab.

In Section 2 we recall a biquaternionic reformulation of the Maxwell system and use it to relate the 1+1 Maxwell system for inhomogeneous media with a hyperbolic Vekua equation. We establish the equivalence between the electromagnetic transmission problem and an initial-value problem for the Vekua equation. In Section 3 we obtain the exact solution of the problem in terms of a couple of the transmutation operators. In Section 4 the analytic approximation of the exact solution is obtained in the case when the incident wave is a partial sum of a trigonometric series. Other types of initial data (and hence modulations) are discussed in Section 5. Section 6 contains several exactly solved examples used as test problems for the resulting numerical method. In Section 7 we formulate some additional conclusions.

2 The 1+1 Maxwell system as a Vekua equation and the problem statement

The algebraic formalism in studying electromagnetic phenomena plays an important role since Maxwell's original treatise in which Hamilton's quaternions were present. In his PhD thesis of 1919 Lanczos wrote the Maxwell system for a vacuum in the form of a single biquaternionic equation. This elegant form of Maxwell' system was rediscovered in several posterior publications, see, e.g., [3].

In [4] (see also [5]) the Maxwell system describing electromagnetic phenomena in isotropic inhomogeneous media was written as a single biquaternionic equation. The Maxwell system has the form

$$\text{rot } \mathbf{H} = \varepsilon \partial_t \mathbf{E} + \mathbf{j}, \quad (2.1)$$

$$\text{rot } \mathbf{E} = -\mu \partial_t \mathbf{H}, \quad (2.2)$$

$$\text{div}(\varepsilon \mathbf{E}) = \rho, \quad (2.3)$$

$$\text{div}(\mu \mathbf{H}) = 0. \quad (2.4)$$

Here ε and μ are real-valued functions of spatial coordinates, \mathbf{E} and \mathbf{H} are real-valued vector fields depending on t and spatial variables, the real-valued scalar function ρ and vector function \mathbf{j} characterize the distribution of sources of the electromagnetic field. The following biquaternionic equation obtained in [4] is equivalent to this system

$$\left(\frac{i}{c} \partial_t - D \right) \mathbf{V} + \mathbf{V} \mathbf{c} + \mathbf{V}^* \mathbf{Z} = \frac{\rho}{\sqrt{\varepsilon}} - i \sqrt{\mu} \mathbf{j}. \quad (2.5)$$

Here

$$\mathbf{c} = \frac{\text{grad } \sqrt{c}}{\sqrt{c}}, \quad \mathbf{Z} = \frac{\text{grad } \sqrt{Z}}{\sqrt{Z}} \quad \text{and} \quad \mathbf{V} = \sqrt{\varepsilon} \mathbf{E} + i \sqrt{\mu} \mathbf{H}$$

where $c = \frac{1}{\sqrt{\varepsilon \mu}}$ is the wave propagation velocity and $Z = \sqrt{\frac{\mu}{\varepsilon}}$ is the the intrinsic impedance of the medium. All magnitudes in bold are understood as purely vectorial biquaternions, and the asterisk denotes the complex conjugation (with respect to the complex imaginary unit i). The operator D is the main quaternionic differential operator introduced by Hamilton himself and sometimes called

the Moisil-Theodoresco operator. It is defined on continuously differentiable biquaternion-valued functions of the real variables x_1 , x_2 and x_3 according to the rule

$$Dq = \sum_{k=1}^3 e_k \partial_k q,$$

where $\partial_k = \frac{\partial}{\partial x_k}$ and e_k are basic quaternionic units.

In [7] with the aid of the representation of the Maxwell system in the form (2.5) it was observed that in the sourceless situation (i.e., ρ and \mathbf{j} are identically zeros) and when all the magnitudes involved are independent of two spatial coordinates, say, x_2 and x_3 , $\varepsilon = \varepsilon(x_1)$ and $\mu = \text{Const}$, the Maxwell system is equivalent to the following Vekua-type equation

$$\partial_{\bar{z}} W - \frac{f'}{2f} \bar{W} = 0 \quad (2.6)$$

where $\partial_{\bar{z}} = \frac{1}{2}(\partial_{\xi} - j\partial_t)$, j is a hyperbolic imaginary unit, $j^2 = 1$ commuting with i , W is a bicomplex-valued function of the real variables ξ and t , $W = u + vj$ and u, v are complex valued (containing the imaginary unit i). The function f is real valued and depends on ξ only. The conjugation with respect to j is denoted by the bar, $\bar{W} = u - vj$.

The Maxwell system in this case can be written in the form

$$\varepsilon(x)\partial_t \mathcal{E} = i\partial_x \mathcal{H}, \quad i\partial_x \mathcal{E} = -\mu\partial_t \mathcal{H} \quad (2.7)$$

where $\mathcal{E} = E_2 + iE_3$, $\mathcal{H} = H_2 + iH_3$, $x = x_1$. The relation between (2.7) and (2.6) involves the change of the independent variable $\xi(x) = \sqrt{\mu} \int_0^x \sqrt{\varepsilon(s)} ds$. The function f in (2.6) is related to ε and μ by the equality $f(\xi) = \sqrt{\tilde{c}(0)}/\sqrt{\tilde{c}(\xi)}$ where and below the tilde means that a function of x is written as a function of ξ , $\tilde{c}(\xi(x)) = c(x)$. The function W is written in terms of E and H as follows

$$W(\xi, t) = \sqrt{\tilde{c}(\xi)} \left(\sqrt{\tilde{\varepsilon}(\xi)} \tilde{\mathcal{E}}(\xi, t) + ij\sqrt{\mu} \tilde{\mathcal{H}} \right). \quad (2.8)$$

Let us consider the problem of a normally incident plane wave transmission through an inhomogeneous medium (see, e.g., [14, Chapter 8]). The electromagnetic field \mathcal{E} and \mathcal{H} is supposed to be known at $x = 0$,

$$\mathcal{E}(0, t) = \mathcal{E}_0(t) \quad \text{and} \quad \mathcal{H}(0, t) = \mathcal{H}_0(t), \quad t \in [\alpha, \beta]. \quad (2.9)$$

We assume \mathcal{E}_0 and \mathcal{H}_0 to be continuously differentiable functions.

The problem (2.7), (2.9) can be reformulated in terms of the function (2.8). Find a solution of (2.6) satisfying the condition

$$W(0, t) = W_0(t) \quad (2.10)$$

where

$$W_0 = \sqrt{c(0)\varepsilon(0)}\mathcal{E}_0 + ij\sqrt{c(0)\mu}\mathcal{H}_0 \quad (2.11)$$

is a given continuously differentiable function.

3 The representation of the solution

First, let us consider the elementary problem

$$w_{\bar{z}} = 0, \quad (3.1)$$

$$w(0, t) = w_0(t). \quad (3.2)$$

The hyperbolic Cauchy-Riemann system (3.1) was studied in several publications (see, e.g., [12], [13], [16] and more recent [10]). Its general solution can be written in the form

$$w(\xi, t) = P^+ \Phi(t + \xi) + P^- \Psi(t - \xi)$$

where Φ and Ψ are arbitrary continuously differentiable scalar functions, $P^\pm = \frac{1}{2}(1 \pm j)$.

For the scalar components of w we introduce the notations

$$\mathcal{R}(w) = u = \frac{1}{2}(w + \bar{w}) \quad \text{and} \quad \mathcal{I}(w) = v = \frac{1}{2j}(w - \bar{w}).$$

When $\xi = 0$ we obtain $w(0, t) = P^+ \Phi(t) + P^- \Psi(t)$. Hence the unique solution of the Cauchy problem (3.1), (3.2) has the form

$$\begin{aligned} w(\xi, t) &= P^+ w_0^+(t + \xi) + P^- w_0^-(t - \xi) \\ &= \frac{1}{2} (w_0^+(t + \xi) + w_0^-(t - \xi) + j (w_0^+(t + \xi) - w_0^-(t - \xi))) \end{aligned} \quad (3.3)$$

where

$$w_0^\pm := \mathcal{R}(w_0) \pm \mathcal{I}(w_0). \quad (3.4)$$

In [10] there was established a relation between solutions of (3.1) and solutions of (2.6). Any solution of (2.6) can be represented in the form

$$W(\xi, t) = T_f [\mathcal{R}(w(\xi, t))] + j T_{1/f} [\mathcal{I}(w(\xi, t))] \quad (3.5)$$

where w is a solution of (3.1), T_f and $T_{1/f}$ are Darboux-associated transmutation operators defined in [9], see also [1] and [10], and applied with respect to the variable ξ . Both operators have the form of Volterra integral operators,

$$T_f u(\xi) = u(\xi) + \int_{-\xi}^{\xi} \mathbf{K}_f(\xi, \tau) u(\tau) d\tau$$

with continuously differentiable kernels. Moreover, the operators T_f and $T_{1/f}$ preserve the value at $\xi = 0$ giving additionally to (3.5) the relation $W(0, t) = w(0, t)$. This together with (3.3) allows us to write down the unique solution of the problem (2.6), (2.10) in the form

$$W(\xi, t) = \frac{1}{2} T_f [W_0^+(t + \xi) + W_0^-(t - \xi)] + \frac{j}{2} T_{1/f} [W_0^+(t + \xi) - W_0^-(t - \xi)]. \quad (3.6)$$

In what follows we use the convenience of this representation and the recent results [10], [11] on the construction of the operators T_f and $T_{1/f}$.

Remark 3.1. The single-wave approximation of the solution of (2.6), (2.10) (see [14, Subsection 8.5.2]) can be obtained from the representation (3.6) by removing integrals from the definition of T_f and $T_{1/f}$ or in other words, replacing T_f and $T_{1/f}$ by the identity operator.

4 Approximation of modulated waves, the simplest initial data

Initial data interesting in practical problems correspond to modulated electromagnetic waves which are represented as partial sums of trigonometric series (other types of initial data are discussed in the next section). In other words, consider initial data of the form

$$\mathcal{E}_0(t) = \sum_{m=-M}^M \alpha_m e^{i(\omega_0+m\omega)t} \quad \text{and} \quad \mathcal{H}_0(t) = \sum_{m=-M}^M \beta_m e^{i(\omega_0+m\omega)t}. \quad (4.1)$$

This leads to a similar form for the initial data W_0 in (2.10),

$$W_0(t) = \sum_{m=-M}^M \gamma_m e^{i(\omega_0+m\omega)t} \quad (4.2)$$

where the bicomplex numbers γ_m are related to $\alpha_m, \beta_m \in \mathbb{C}$ as follows

$$\gamma_m = \sqrt{c(0)} \left(\sqrt{\varepsilon(0)} \alpha_m + i j \sqrt{\mu} \beta_m \right).$$

Due to (3.6), the unique solution of the problem (2.6), (2.10) with W_0 given by (4.2) can be written in the form

$$\begin{aligned} W(\xi, t) = \frac{1}{2} \left(\sum_{m=-M}^M e^{i(\omega_0+m\omega)t} \left(\gamma_m^+ T_f \left[e^{i(\omega_0+m\omega)\xi} \right] + \gamma_m^- T_f \left[e^{-i(\omega_0+m\omega)\xi} \right] \right) \right. \\ \left. + j \sum_{m=-M}^M e^{i(\omega_0+m\omega)t} \left(\gamma_m^+ T_{1/f} \left[e^{i(\omega_0+m\omega)\xi} \right] - \gamma_m^- T_{1/f} \left[e^{-i(\omega_0+m\omega)\xi} \right] \right) \right) \end{aligned} \quad (4.3)$$

where $\gamma_m^\pm = \sqrt{c(0)} \left(\sqrt{\varepsilon(0)} \alpha_m \pm i j \sqrt{\mu} \beta_m \right)$.

Although the explicit form of the operators T_f and $T_{1/f}$ is usually unknown, in [11] it was shown how their kernels can be approximated by means of generalized wave polynomials. In particular, for the images of the functions $e^{\pm i(\omega_0+m\omega)\xi}$ the following approximate representations are valid

$$\begin{aligned} T_f \left[e^{\pm i(\omega_0+m\omega)\xi} \right] \cong e^{\pm i(\omega_0+m\omega)\xi} + 2 \sum_{n=0}^N a_n \sum_{\text{even } k=0}^n \binom{n}{k} \varphi_{n-k}(\xi) \int_0^\xi \tau^k \cos(\omega_0 + m\omega) \tau \, d\tau \\ \pm 2i \sum_{n=1}^N b_n \sum_{\text{odd } k=1}^n \binom{n}{k} \varphi_{n-k}(\xi) \int_0^\xi \tau^k \sin(\omega_0 + m\omega) \tau \, d\tau \end{aligned} \quad (4.4)$$

and

$$\begin{aligned} T_{1/f} \left[e^{\pm i(\omega_0+m\omega)\xi} \right] \cong e^{\pm i(\omega_0+m\omega)\xi} - 2 \sum_{n=0}^N b_n \sum_{\text{even } k=0}^n \binom{n}{k} \psi_{n-k}(\xi) \int_0^\xi \tau^k \cos(\omega_0 + m\omega) \tau \, d\tau \\ \mp 2i \sum_{n=1}^N a_n \sum_{\text{odd } k=1}^n \binom{n}{k} \psi_{n-k}(\xi) \int_0^\xi \tau^k \sin(\omega_0 + m\omega) \tau \, d\tau. \end{aligned} \quad (4.5)$$

Here all integrals are easily calculated in a closed form, the functions φ_n and ψ_n are defined as follows. Consider two sequences of recursive integrals

$$X^{(0)}(x) \equiv 1, \quad X^{(n)}(x) = n \int_0^x X^{(n-1)}(s) (f^2(s))^{(-1)^n} ds, \quad n = 1, 2, \dots \quad (4.6)$$

and

$$\tilde{X}^{(0)} \equiv 1, \quad \tilde{X}^{(n)}(x) = n \int_0^x \tilde{X}^{(n-1)}(s) (f^2(s))^{(-1)^{n-1}} ds, \quad n = 1, 2, \dots \quad (4.7)$$

The two families of functions $\{\varphi_k\}_{k=0}^\infty$ and $\{\psi_k\}_{k=0}^\infty$ are constructed according to the rules

$$\varphi_k(x) = \begin{cases} f(x)X^{(k)}(x), & k \text{ odd}, \\ f(x)\tilde{X}^{(k)}(x), & k \text{ even}, \end{cases} \quad (4.8)$$

and

$$\psi_k(x) = \begin{cases} \frac{\tilde{X}^{(k)}(x)}{f(x)}, & k \text{ odd}, \\ \frac{X^{(k)}(x)}{f(x)}, & k \text{ even}. \end{cases} \quad (4.9)$$

Finally, the coefficients a_n and b_n are obtained by solving an approximation problem described in [11].

As an important feature of the representations (4.4) and (4.5) in [11] there were obtained estimates of their accuracy uniform with respect to the parameter $(\omega_0 + m\omega)$.

Note that the direct evaluation of expressions (4.4) and (4.5) requires $O(N^2)$ algebraic operations for each pair of m and ξ leading to the computation complexity $O(MN^2)$ for the evaluation of (4.3). The change of summation order in (4.4) and (4.5) allows one to evaluate (4.3) with the computation complexity of $O(NM + N^2)$. For example, consider

$$\begin{aligned} & \sum_{n=0}^N a_n \sum_{\text{even } k=0}^n \binom{n}{k} \varphi_{n-k}(\xi) \int_0^\xi \tau^k \cos(\omega_0 + m\omega)\tau d\tau \\ &= \sum_{\text{even } k=0}^N \left(\sum_{n=k}^N a_n \binom{n}{k} \varphi_{n-k}(\xi) \right) \int_0^\xi \tau^k \cos(\omega_0 + m\omega)\tau d\tau. \end{aligned}$$

The coefficients $\sum_{n=k}^N a_n \binom{n}{k} \varphi_{n-k}(\xi)$ can be precomputed once for every ξ in $O(N^2)$ operations and the outer sum requires $O(N)$ operations.

5 Other types of initial data

The proposed method is not restricted exclusively to initial data which can be represented or closely approximated by (4.1). When one can efficiently calculate (at least numerically) the indefinite integrals

$$\int \tau^k W_0^+(\tau) d\tau \quad \text{and} \quad \int \tau^k W_0^-(\tau) d\tau, \quad k = 0, \dots, N,$$

the approximations of the transmutation operators can be used. One of the examples of such initial data arises in digital signal transmission. For widely used modulations like phase-shift keying or QAM, the transmitted signal can be represented as

$$s(t) = \sum_{n=0}^M [c_n \cos \omega_0 t + s_n \sin \omega_0 t] \mathbf{1}_{[nf_s, (n+1)f_s)}(t),$$

where $\mathbf{1}_{[a,b)}$ is the characteristic function of the interval $[a, b)$, ω_0 is the carrier frequency, f_s is the symbol rate and the coefficients s_n and c_n encode transmitted information. Other examples include Gaussian RF pulses and linear frequency modulation (also known as chirp modulation) used for radars. We refer the reader to [15] for further details.

Returning to (3.6), consider $T_f[W_0^+(t+\xi)]$. In [11] the following approximation was constructed

$$\begin{aligned} T_f[W_0^+(t+\xi)](\xi) &\cong W_0^+(t+\xi) + \sum_{n=0}^N a_n \sum_{\text{even } k=0}^n \binom{n}{k} \varphi_{n-k}(\xi) \int_{-\xi}^{\xi} \tau^k W_0^+(t+\tau) d\tau \\ &+ \sum_{n=1}^N b_n \sum_{\text{odd } k=1}^n \binom{n}{k} \varphi_{n-k}(\xi) \int_{-\xi}^{\xi} \tau^k W_0^+(t+\tau) d\tau. \end{aligned} \quad (5.1)$$

We have

$$\int_{-\xi}^{\xi} \tau^k W_0^+(t+\tau) d\tau = \int_{t-\xi}^{t+\xi} (z-t)^k W_0^+(z) dz = \sum_{\ell=0}^k \binom{k}{\ell} (-1)^{k-\ell} t^{k-\ell} \int_{t-\xi}^{t+\xi} z^{\ell} W_0^+(z) dz,$$

hence reordering terms in (5.1) one obtains

$$\begin{aligned} T_f[W_0^+(t+\xi)] &\cong W_0^+(t+\xi) + \sum_{\ell=0}^N \left(\int_{t-\xi}^{t+\xi} z^{\ell} W_0^+(z) dz \right) \\ &\times \sum_{n=\ell}^N \left\{ a_n \sum_{\text{even } k \geq \ell}^n + b_n \sum_{\text{odd } k \geq \ell}^n \right\} (-1)^{k-\ell} \binom{n}{k} \binom{k}{\ell} \varphi_{n-k}(\xi) t^{k-\ell}, \end{aligned} \quad (5.2)$$

an expression which can be easily evaluated once the indefinite integrals of $z^{\ell} W_0^+(z)$ are known. Similarly,

$$\begin{aligned} T_f[W_0^-(t-\xi)] &\cong W_0^-(t-\xi) + \sum_{\ell=0}^N \left(\int_{t-\xi}^{t+\xi} z^{\ell} W_0^-(z) dz \right) \\ &\times \sum_{n=\ell}^N \left\{ a_n \sum_{\text{even } k \geq \ell}^n + b_n \sum_{\text{odd } k \geq \ell}^n \right\} (-1)^{\ell} \binom{n}{k} \binom{k}{\ell} \varphi_{n-k}(\xi) t^{k-\ell}. \end{aligned} \quad (5.3)$$

To obtain approximations for $T_{1/f}[W_0^{\pm}(t \pm \xi)]$ one can change in (5.2) and (5.3) the coefficients a_n and b_n by $-b_n$ and $-a_n$ respectively and change functions ϕ_n by ψ_n .

6 Examples and numerical tests

Example 6.1. Let us consider the system (2.7) with the permittivity of the form

$$\varepsilon(x) = (\alpha x + \beta)^{-2}, \quad (6.1)$$

where α and β are some real numbers, such that $\alpha x + \beta$ does not vanish on the interval of interest and $\varepsilon > 0$. Then $\xi = \sqrt{\mu} \int_0^x \sqrt{\varepsilon(s)} ds = \frac{\sqrt{\mu}}{\alpha} \log \frac{\alpha x + \beta}{\beta}$. Hence

$$x = \frac{\beta}{\alpha} \left(e^{\frac{\alpha \xi}{\sqrt{\mu}}} - 1 \right) \quad \text{and} \quad \tilde{\varepsilon}(\xi) = \frac{1}{\beta^2} e^{-\frac{2\alpha \xi}{\sqrt{\mu}}}, \quad \tilde{c}(\xi) = \frac{\beta}{\sqrt{\mu}} e^{\frac{\alpha \xi}{\sqrt{\mu}}}, \quad f(\xi) = e^{-\frac{\alpha \xi}{2\sqrt{\mu}}}.$$

In this case the Vekua equation (2.6) has the form

$$\partial_{\bar{z}} W + \gamma \bar{W} = 0 \quad (6.2)$$

where the coefficient γ is constant, $\gamma = \alpha / (4\sqrt{\mu})$.

The Vekua equation (2.6) is a special case of the main Vekua equation $\partial_{\bar{z}} W - \frac{f_{\bar{z}}}{f} \bar{W} = 0$ (see [8], [6]). For its solutions one has that $\mathcal{R}(W)$ satisfies the equation $(\square - q)u = 0$ where $\square := \partial_{\xi}^2 - \partial_t^2$ and $q = \square f / f$, and $\mathcal{I}(W)$ satisfies the equation $(\square - r)v = 0$ where $r = \square(f^{-1}) / (f^{-1})$. The relation between $\mathcal{R}(W)$ and $\mathcal{I}(W)$ is akin to the relation between harmonic conjugate functions and can be found in [8], [6].

In the case of equation (6.2) $q = r = \alpha^2 / (4\mu)$. Consider a solution u of the equation $(\square - \frac{\alpha^2}{4\mu})u = 0$ in the form $u(\xi, t) = e^{2i\gamma t}(A\xi + B)$ where A and B are arbitrary real constants. Then a solution W of (6.2) such that $\mathcal{R}(W) = u$ can be chosen in the form (see [8], [6])

$$W(\xi, t) = e^{2i\gamma t} \left((1 - ij)(A\xi + B) - ij \frac{A}{2\gamma} \right). \quad (6.3)$$

This leads to the following solution of the Maxwell system

$$\tilde{\mathcal{E}}(\xi, t) = \frac{1}{\sqrt{\tilde{c}(\xi)\tilde{\varepsilon}(\xi)}} \mathcal{R}(W(\xi, t)) = \frac{e^{2i\gamma t}}{\sqrt{\tilde{c}(\xi)\tilde{\varepsilon}(\xi)}} (A\xi + B)$$

and

$$\tilde{\mathcal{H}}(\xi, t) = -\frac{i}{\sqrt{\tilde{c}(\xi)\mu}} \mathcal{I}(W(\xi, t)) = -\frac{e^{2i\gamma t}}{\sqrt{\tilde{c}(\xi)\mu}} \left(A\xi + B + \frac{A}{2\gamma} \right)$$

or in terms of the variables x and t ,

$$\mathcal{E}(x, t) = \sqrt[4]{\mu} \sqrt{\alpha x + \beta} e^{\frac{i\alpha t}{2\sqrt{\mu}}} \left(\frac{\sqrt{\mu}}{\alpha} A \log \frac{\alpha x + \beta}{\beta} + B \right) \quad (6.4)$$

and

$$\mathcal{H}(x, t) = -\frac{e^{\frac{i\alpha t}{2\sqrt{\mu}}}}{\sqrt[4]{\mu} \sqrt{\alpha x + \beta}} \left(\frac{\sqrt{\mu}}{\alpha} A \log \frac{\alpha x + \beta}{\beta} + B + \frac{2A\sqrt{\mu}}{\alpha} \right). \quad (6.5)$$

Thus the functions (6.4) and (6.5) are the exact solutions of the Maxwell system (2.7) with $\varepsilon(x)$ given by (6.1) and with the initial conditions

$$\mathcal{E}(0, t) = \sqrt[4]{\mu} \sqrt{\beta} e^{\frac{i\alpha t}{2\sqrt{\mu}}} B \quad \text{and} \quad \mathcal{H}(0, t) = -\frac{e^{\frac{i\alpha t}{2\sqrt{\mu}}}}{\sqrt[4]{\mu} \sqrt{\beta}} \left(B + \frac{2A\sqrt{\mu}}{\alpha} \right).$$

Below we present the results of numerical solution of the same system by the proposed method.

For the numerical experiment we considered an interval $[0, 5]$ for both x and t and took $\alpha = 2$, $\beta = 1$, $\mu = 1$ and $A = 1$, $B = 3$. The initial condition in this example corresponds to the case $M = 0$ and $\omega_0 = 2\gamma$ in (4.2). The permittivity $\varepsilon(x)$ was approximated on uniform mesh of 5001 points. The new variable ξ was obtained by the modified 6 point Newton-Cottes integration formula, see [2] for details. The same integration formula was used for calculation of all other integrals, i.e., for computation of (4.6) and (4.7) and for the integration of the potential q required to obtain approximations (4.4) and (4.5). Note that the integration with respect to the variable ξ requires integration over a non-uniform mesh, however such inconvenience can be easily avoided observing that $\int_0^{\xi(x)} \tilde{g}(\xi) d\xi = \int_0^x g(s) \xi'(s) ds = \int_0^x g(s) \sqrt{\mu \varepsilon(s)} ds$ for any function $g(x) = \tilde{g}(\xi(x))$.

All calculations were performed using Matlab 2012 in the machine precision. The exact expressions were used only for the function $\varepsilon(x)$ and its derivatives, all other functions involved were computed numerically.

The developed program found the optimal value of N for the approximations (4.4) and (4.5) to be $N = 14$. The computation time required was 0.4 seconds. On Figure 1 we show the graphs of the absolute errors of the computed $\mathcal{E}(x, t)$ and $\mathcal{H}(x, t)$.

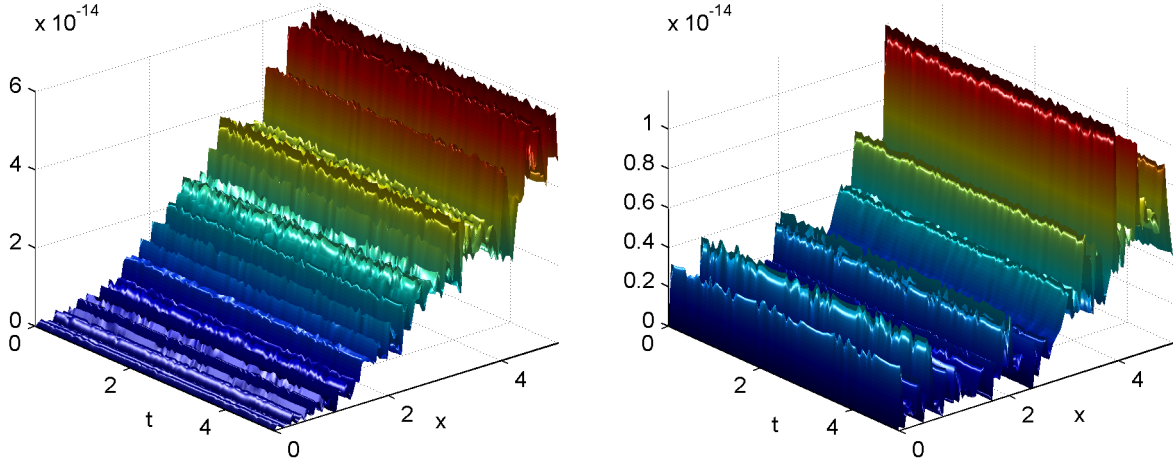


Figure 1: Graphs of the absolute errors of $\mathcal{E}(x, t)$ (on the left) and $\mathcal{H}(x, t)$ (on the right) from Example 6.1.

Example 6.2. Let us consider the same parameters of the medium as in Example 6.1 but instead of choosing a solution of (6.2) in the form (6.3) we take the solution

$$W(\xi, t) = A e^{i\Omega t} \left(e^{D\xi} + \frac{D+C}{D-C} e^{-D\xi} + \frac{2ij\Omega}{D-C} \sinh D\xi \right). \quad (6.6)$$

Here A and Ω are arbitrary constants, $C = \frac{\alpha}{2\sqrt{\mu}}$, $D = i\sqrt{\Omega^2 - C^2}$. It is obtained similarly to (6.3) starting with a solution u of the equation $\left(\square - \frac{\alpha^2}{4\mu} \right) u = 0$ in the form $u(\xi, t) = A e^{i\Omega t} (e^{D\xi} + \frac{D+C}{D-C} e^{-D\xi})$.

We have then that

$$\mathcal{E}(x, t) = A \sqrt[4]{\mu} \sqrt{\alpha x + \beta} e^{i\Omega t} \left(\left(\frac{\alpha x + \beta}{\beta} \right)^{\frac{D\sqrt{\mu}}{\alpha}} + \frac{D+C}{D-C} \left(\frac{\alpha x + \beta}{\beta} \right)^{-\frac{D\sqrt{\mu}}{\alpha}} \right)$$

and

$$\mathcal{H}(x, t) = \frac{A}{D - C} \frac{\Omega e^{i\Omega t}}{\sqrt[4]{\mu} \sqrt{\alpha x + \beta}} \left(\left(\frac{\alpha x + \beta}{\beta} \right)^{\frac{D\sqrt{\mu}}{\alpha}} - \left(\frac{\alpha x + \beta}{\beta} \right)^{-\frac{D\sqrt{\mu}}{\alpha}} \right)$$

satisfy the Maxwell system (2.7) with the permittivity (6.1) and the initial conditions

$$\mathcal{E}(0, t) = \frac{2AD}{D - C} \sqrt[4]{\mu} \sqrt{\beta} e^{i\Omega t} \quad \text{and} \quad \mathcal{H}(0, t) = 0.$$

For the numerical calculation we considered the same values of the parameters α , β , μ as in Example 6.1 and took the interval $[0, 6]$ for both x and t . For the initial condition we took the sum of four terms, each of the form (6.6) having $\Omega_1 = -\Omega_2 = C + 1$, $\Omega_3 = -\Omega_4 = C + 2$. Since the expression (6.6) for $\xi = 0$ reduces to $W(0, t) = \frac{2AD}{D - C} e^{i\Omega t}$, we took $A_i = \frac{D_i - C}{D_i}$, $i = 1, \dots, 4$ and obtained initial conditions $W_0^\pm(t) = 4 \cos(C + 1)t + 4 \cos(C + 2)t$.

For this example the optimal N was equal to 13 and the whole computation time was 0.3 seconds. On Figure 2 we present the graphs of the initial condition and the computed $\mathcal{E}(x, t)$. The absolute errors of the computed $\mathcal{E}(x, t)$ and $\mathcal{H}(x, t)$ were less than $1.1 \cdot 10^{-13}$ and $9 \cdot 10^{-15}$ respectively.

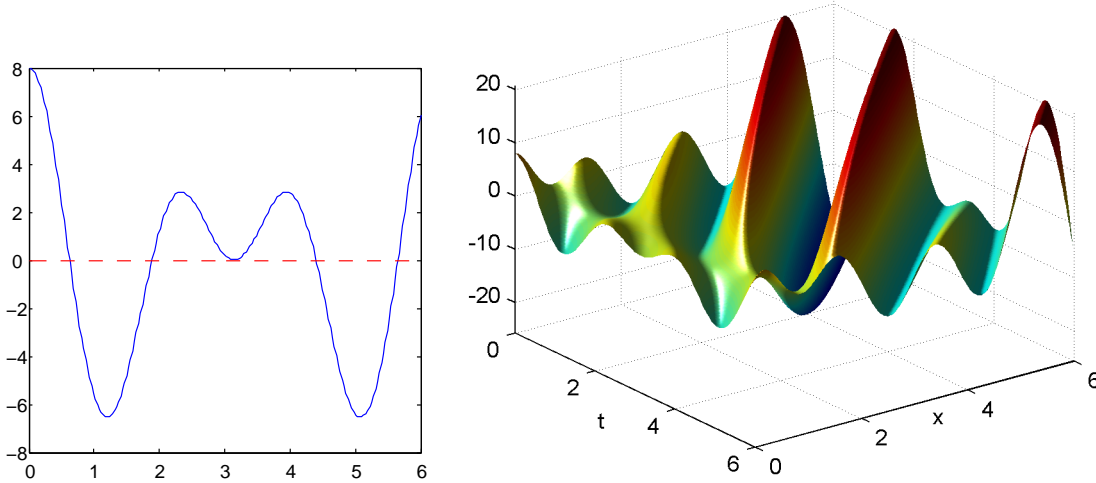


Figure 2: Graphs of the initial data $W_0^\pm(t)$ (on the left, real part in solid blue line, imaginary part in dashed red line) and of $\mathcal{E}(x, t)$ (on the right) from Example 6.2.

Example 6.3. For this example let us consider the system (2.7) with the permittivity of the form $\varepsilon(x) = (\alpha x + \beta)^{-2+2\ell}$ where $\ell \neq 0$ and $\alpha, \beta \in \mathbb{R}$ are such that $\alpha x + \beta > 0$ on the interval of interest. Then $\xi(x) = \sqrt{\mu} \int_0^x \sqrt{\varepsilon(s)} ds = \frac{\sqrt{\mu}}{\alpha \ell} ((\alpha x + \beta)^\ell - \beta^\ell)$. Hence

$$x = \frac{\beta}{\alpha} \left(\left(1 + \frac{\alpha \ell \xi}{\sqrt{\mu} \beta^\ell} \right)^{1/\ell} - 1 \right), \quad \tilde{c}(\xi) = \frac{1}{\sqrt{\mu}} \left(\frac{\alpha \ell \xi}{\sqrt{\mu}} + \beta^\ell \right)^{(1-\ell)/\ell} \quad \text{and} \quad \tilde{f}(\xi) = \frac{1}{\left(1 + \frac{\alpha \ell}{\sqrt{\mu} \beta^\ell} \xi \right)^{\frac{1-\ell}{2\ell}}}.$$

In [9] we show how one can construct the transmutation operators T_f and $T_{1/f}$ when $f = (1 + \xi)^n$, $n \in \mathbb{Z}$. The procedure can be easily generalized for functions of the form $f(\xi) = (1 + c\xi)^n$,

$n \in \mathbb{Z}$. Hence for values of ℓ of the form $\ell = \frac{1}{1-2n}$, $n \in \mathbb{Z}$ one can explicitly construct the pair of transmutation operators T_f and $T_{1/f}$ and obtain the solution of (3.6).

For the numerical experiment we took $\ell = 1/5$, $\alpha = 5$, $\beta = 1$ and $\mu = 1$. For such parameters the function \tilde{f} is equal to $\frac{1}{(1+\xi)^2}$ and the integral kernels of the transmutation operators T_f and $T_{1/f}$ are given by (see [9])

$$\mathbf{K}_{\tilde{f}}(\xi, t) = \frac{(3t-1)(\xi+1)^2 - 3(t-1)^2(t+1)}{4(\xi+1)^2} \quad \text{and} \quad \mathbf{K}_{1/\tilde{f}}(\xi, t) = \frac{3\xi^2 + 6\xi + 4 - 3t^2 + 2t}{4(\xi+1)}.$$

We considered the interval $[0, 2]$ for x and the interval $[-2, 2]$ for t and took the Gaussian pulse $W_0^+(t) = 0$, $W_0^-(t) = e^{-4t^2}$ as the initial condition. For such initial condition the expression (3.6) can be evaluated in the terms of **erf** function. The approximate solution was computed using (5.2) and (5.3). We used 2001 points to represent the permittivity $\varepsilon(x)$. The formal powers were computed as it was explained in Example 6.1 while to evaluate the indefinite integrals $\int z^k W_0^\pm(z) dz$, $k = 0, \dots, N$ we approximated the integrands as splines and used the function **fnint** from Matlab. The main reason for such choice is that despite the uniform mesh was taken for both x and t , the resulting mesh for ξ may not be uniform leading to rather large set of values which can be taken by $t - \xi$ and $t + \xi$. All computations were performed in machine precision in Matlab 2012. On Figure 3 we show the obtained graphs of $\mathcal{E}(x, t)$ and $\mathcal{H}(x, t)$. The absolute errors of the computed solutions were less than $1.1 \cdot 10^{-14}$ and $2.6 \cdot 10^{-14}$, respectively.

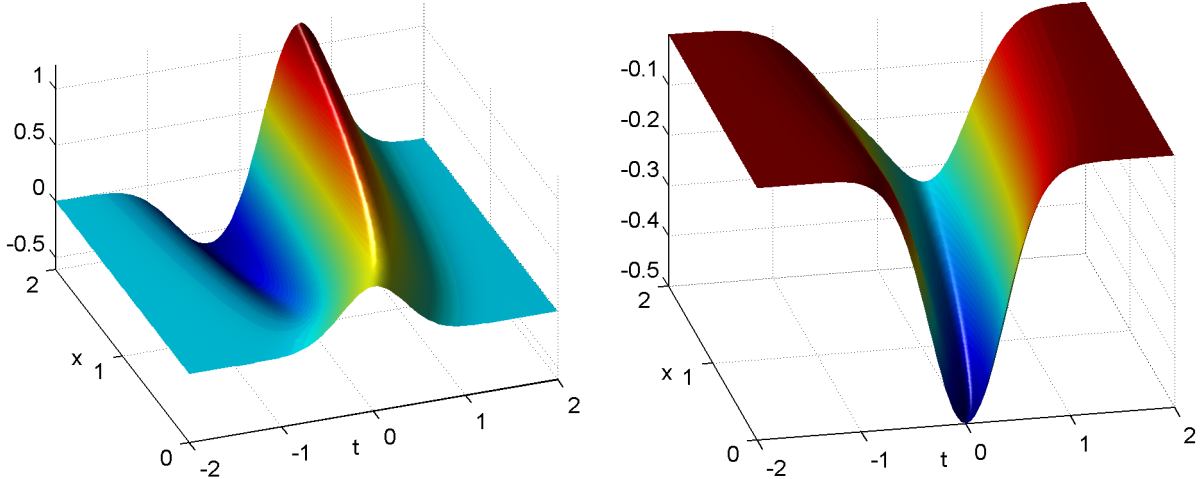


Figure 3: Graphs of $\mathcal{E}(x, t)$ (on the left) and of $\mathcal{H}(x, t)$ (on the right) from Example 6.3.

7 Conclusions

A method for solving the problem of electromagnetic wave propagation through an inhomogeneous medium is developed. It is based on a simple transformation of the Maxwell system into a hyperbolic Vekua equation and on the solution of this equation by means of approximate transmutation operators. In spite of elaborate mathematical results which are behind of the proposed method, the final representations for approximate solutions of the electromagnetic problem have a sufficiently simple form, their numerical implementation is straightforward and can use standard routines of such packages as Matlab.

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